[7] Clearly N is connected, because every point of N can be joined to A by a simple closed curve which is a subset of N.

[8] Cf. Moore, R. L., *Math. Zeit.*, **15**, 1922 (254–260).

[9] Cf. an abstract of a paper by the latter in the *Bull. Amer. Math. Soc.*, **32**, 1926 (14).

[10] Cf. my paper, "Concerning Continua in the Plane," *loc. cit.*, theorem 24.

[11] This theorem is to the effect that the set of all the cut points of a continuous curve cannot contain more than finite number of mutually exclusive continua of diameter greater than a given positive number, *loc. cit.*, theorem 15.

[12] A two-way continuous curve is a continuous curve having the property that every two of its points can be joined by two arcs of that curve neither of which is a subset of the other. Cf. my paper, "Two-Way Continuous Curves," *Bull. Amer. Math. Soc.*, **32**, 1926 (659–663).

[13] Moore, R. L., *Trans. Amer. Math. Soc.*, **21**, 1920 (345).

[14] Moore, R. L., *Fund. Math.*, **6**, 1925 (212).

DISPLACEMENTS IN A GEOMETRY OF PATHS WHICH CARRY PATHS INTO PATHS

BY L. P. EISENHART AND M. S. KNEBELMAN

PRINCETON UNIVERSITY

Communicated January 8, 1927

1. In a space of n dimensions, where $x^1, x^2, \ldots, x^n$ are general coordinates, the equations of the paths are

$$\frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) - \frac{dx^i}{dt}\left(\frac{d^2x^j}{dt^2} + \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) = 0, \quad (1.1)$$

where t is a general parameter and Γ^i_{jk} are functions of the x's such that $\Gamma^i_{jk} = \Gamma^i_{kj}$.

For a particular path, that is, an integral curve of equations (1.1) we have

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \varphi\frac{dx^i}{dt},$$

where φ is a determinate function of t. If we define a parameter s by

$$\frac{ds}{dt} = e^{\int \varphi dt},$$

the above equations become

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \quad (1.2)$$

Thus the parameter s, which we call the *affine parameter* of the path, is the analogue of the arc of a geodesic in a Riemannian space.

When we effect a general transformation of coördinates x^i into coördinates x'^i and require that s is not changed but that we get equations of the form (1.2) with x^i and Γ^i_{jk} replaced by x'^i and Γ'^i_{jk}, we get

$$\frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Gamma'^\alpha_{\beta\gamma}\frac{\partial x'^\beta}{\partial x^j}\frac{\partial x'^\gamma}{\partial x^k} - \Gamma^i_{jk}\frac{\partial x'^\alpha}{\partial x^i} = 0, \tag{1.3}$$

where the indices take the values 1 to n and the summation convention of a repeated index is used.[1] Functions Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ related as in (1.3) are called the coefficients of an affine connection in the two coördinate systems and the geometry based upon them, an affine geometry of paths.

2. When we do not require that s is not changed by the transformation, and consequently that equations (1.1) are transformed into analogous ones in the primes (since t is a general parameter, there is no loss of generality in using it in both sets of equations), we obtain

$$\left(\frac{\partial x'^\alpha}{\partial x^i}A^\beta_{jk} - \frac{\partial x'^\beta}{\partial x^i}A^\alpha_{jk}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,$$

where

$$A^\alpha_{jk} = \frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Gamma'^\alpha_{\gamma\delta}\frac{\partial x'^\gamma}{\partial x^j}\frac{\partial x'^\delta}{\partial x^k} - \Gamma^i_{jk}\frac{\partial x'^\alpha}{\partial x^i}. \tag{2.1}$$

Since the above condition must be satisfied for all the paths, we must have

$$\frac{\partial x'^\alpha}{\partial x^i}A^\beta_{jk} - \frac{\partial x'^\beta}{\partial x^i}A^\alpha_{jk} + \frac{\partial x'^\alpha}{\partial x^j}A^\beta_{ki} - \frac{\partial x'^\beta}{\partial x^j}A^\alpha_{ki} + \frac{\partial x'^\alpha}{\partial x^k}A^\beta_{ij} - \frac{\partial x'^\beta}{\partial x^k}A^\alpha_{ij} = 0. \tag{2.2}$$

Multiplying by $\dfrac{\partial x^i}{\partial x'^\alpha}$ and summing for i and α, we obtain in consequence of (2.1)

$$(n+1)A^\beta_{jk} = \frac{\partial x'^\beta}{\partial x^j}\left(\frac{\partial \log \Delta}{\partial x^k} + \Gamma'^\alpha_{\alpha\gamma}\frac{\partial x'^\gamma}{\partial x^k} - \Gamma^i_{ik}\right) + \frac{\partial x'^\beta}{\partial x^k}\left(\frac{\partial \log \Delta}{\partial x^j} + \Gamma'^\alpha_{\alpha\gamma}\frac{\partial x'^\gamma}{\partial x^j} - \Gamma^i_{ij}\right), \tag{2.3}$$

where Δ is the jacobian $\left|\dfrac{\partial x'^\alpha}{\partial x^i}\right|$. When the expressions (2.3) are substituted in (2.2), the latter are satisfied identically. Hence the conditions upon the Γ's and Γ''s are given by combining (2.1) and (2.3), namely,

$$\frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Pi'^\alpha_{\beta\gamma} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x'^\gamma}{\partial x^k} - \Pi^i_{jk} \frac{\partial x'^\alpha}{\partial x^i} - \frac{\partial \theta}{\partial x^j} \frac{\partial x'^\alpha}{\partial x^k} - \frac{\partial \theta}{\partial x^k} \frac{\partial x'^\alpha}{\partial x^j} = 0, \quad (2.4)$$

where

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+1}\left(\delta^i_j \Gamma^h_{hk} + \delta^i_k \Gamma^h_{hj}\right) \quad (2.5)$$

and

$$\theta = \frac{1}{n+1} \log \Delta . \quad (2.6)$$

If there is not a change of coördinates, but merely of the affine parameters, it follows that $\Pi^i_{jk} = \overline{\Pi}^i_{jk}$, where $\overline{\Pi}^i_{jk}$ is formed as in (2.5) from a set of $\overline{\Gamma}$'s. These functions Π^i_{jk} were first obtained by T. Y. Thomas in another manner[2] and were called the *coefficients of projective connections* and the geometry of such connections projective geometry of paths.

From (1.3) it follows that

$$\Gamma^i_{ij} = \Gamma'^\alpha_{\alpha\beta} \frac{\partial x'^\beta}{\partial x^j} + (n+1) \frac{\partial \theta}{\partial x^j}. \quad (2.7)$$

Moreover, when (2.7) were satisfied, equations (2.4) reduce to (1.3).

3. The equations of the preceding section may be used to define transformations of an affinely connected manifold into itself in such a manner that paths are transformed into paths. In this case the conditions of the problem are given by (2.4) on the assumption that Γ'^i_{jk} are the same functions of the x''s as the Γ's with the same indices are of the x's. If the transformations form a continuous group of r parameters, they may be considered as generated by r infinitesimal transformations. Accordingly we consider the question of infinitesimal transformations.

Consider an infinitesimal transformation defined by

$$x'^i = x^i + \xi^i \delta u, \quad (3.1)$$

where the ξ's are functions of the x's and δu is an infinitesimal. Since by hypothesis the Γ's are the same functions of the x's as the corresponding Γ''s are of the x's, the same is true of the Π's and Π''s.

Accordingly we have

$$\Pi'^i_{jk} = \Pi^i_{jk} + \frac{\partial \Pi^i_{jk}}{\partial x^h} \xi^h \delta u, \quad (3.2)$$

neglecting infinitesimals of the second and higher orders; this will be done in what follows. From (3.1) it follows that the determinant Δ of the transformation is given by

$$\Delta = 1 + \frac{\partial \xi^h}{\partial x^h}\,\delta u$$

and consequently from (2.6)

$$\frac{\partial \theta}{\partial x^i} = \frac{1}{n+1}\,\frac{\partial^2 \xi^h}{\partial x^h \partial x^i}\,\delta u\,. \tag{3.3}$$

When these values are substituted in (2.4), we obtain, on neglecting the multiplier δu,

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + \Pi^\alpha_{ik}\frac{\partial \xi^k}{\partial x^j} + \Pi^\alpha_{jk}\frac{\partial \xi^k}{\partial x^i} + \xi^h\frac{\partial \Pi^\alpha_{ij}}{\partial x^h} - \Pi^k_{ij}\frac{\partial \xi^\alpha}{\partial x^k}$$
$$- \frac{1}{n+1}\left(\delta^\alpha_j \frac{\partial^2 \xi^h}{\partial x^h \partial x^i} + \delta^\alpha_i \frac{\partial^2 \xi^h}{\partial x^h \partial x^j}\right) = 0.$$

Because of (2.5) these equations become

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{ik}\frac{\partial \xi^k}{\partial x^j} + \Gamma^\alpha_{jk}\frac{\partial \xi^k}{\partial x^i} + \xi^h\frac{\partial \Gamma^\alpha_{ij}}{\partial x^h} - \Gamma^k_{ij}\frac{\partial \xi^\alpha}{\partial x^k} = \delta^\alpha_j \varphi_i + \delta^\alpha_i \varphi_j\,, \tag{3.4}$$

where

$$\varphi_i = \frac{1}{n+1}\left(\frac{\partial^2 \xi^h}{\partial x^h \partial x^i} + \Gamma^h_{hk}\frac{\partial \xi^k}{\partial x^i} + \xi^h \frac{\partial \Gamma^k_{ki}}{\partial x^h}\right). \tag{3.5}$$

We remark that when the first term of (3.4) is contracted for α and j, we get $(n+1)\varphi_i$. Equations (3.4) may be written in the form

$$\xi^h_{,ij} = \xi^k B^h_{ijk} + \delta^h_j \varphi_i + \delta^h_i \varphi_j\,, \tag{3.6}$$

where one or more indices preceded by a comma indicate covariant differentiation with respect to the Γ's and B^h_{ijk} are the components of the affine curvature tensor. Contracting for h and i, we have

$$\xi^i_{,ij} - \xi^h B^i_{ijh} = (n+1)\varphi_j\,. \tag{3.7}$$

If the affine parameter s is unaltered by the infinitesimal transformation, it follows from (2.7) and (3.3) that φ_i as defined by (3.5) are zero. In this case equations (3.6) become

$$\xi^h_{,ij} = \xi^k B^h_{ijk}\,. \tag{3.8}$$

According as ξ^i's satisfy (3.6) or (3.8) we say that equations (3.1) define a *projective or affine infinitesimal displacement* of the space into itself.

4. Suppose that we have a solution ξ^i of (3.6) and (3.7), and that the coördinates x^i are chosen so that in this coördinate system[3]

$$\xi^1 = 1, \quad \xi^\alpha = 0 \qquad (\alpha = 2, \ldots, n). \tag{4.1}$$

In this case equations (3.6) reduce to

$$\frac{\partial \Gamma^i_{jk}}{\partial x^1} = \delta^i_j \varphi_k + \delta^i_k \varphi_j \tag{4.2}$$

from which we have

$$\frac{\partial \Pi^i_{jk}}{\partial x^1} = 0. \tag{4.3}$$

In consequence of these equations we have:

When an affinely connected space admits a projective or affine infinitesimal displacement of the space into itself, the finite group G_1 generated by it transforms paths into paths.

In fact, for the chosen coördinate system the equations of the finite group are

$$x'^1 = x^1 + a, \qquad x'^\alpha = x^\alpha \qquad (\alpha = 2, \ldots, n)$$

where a is a parameter. For this transformation equations (2.4) reduce to $\Pi'^i_{jk} = \Pi^i_{jk}$. In consequence of (4.3) this condition is satisfied for a projective displacement. For an affine displacement we have from (4.2) that Γ^i_{jk} is independent of x^1 so that the theorem follows in this case also. Moreover, we have shown incidentally that:

The most general affinely connected manifold which admits a finite group G_1 of affine displacements into itself is given by taking for Γ^i_{jk} functions of $(n-1)$ of the coördinates.

5. The conditions of integrability of equations (3.6) are

$$\xi^h B^i_{jkl,h} - \xi^i_{,h} B^h_{jkl} + \xi^h_{,j} B^i_{hkl} + \xi^h_{,k} B^i_{jhl} + \xi^h_{,l} B^i_{jkh} = \delta^i_l \varphi_{j,k} - \delta^i_k \varphi_{j,l} + \delta^i_j (\varphi_{l,k} - \varphi_{k,l}).$$

By means of these equations it can be shown that if ξ^i_1 and ξ^i_2 are solutions of (3.6) and $X_1 f$ and $X_2 f$ are the corresponding generators, then the Poisson operator $(X_1 X_2 - X_2 X_1) f$ defines another projective or affine displacement, according as φ_i are different from zero, or are zero.

For a Riemannian geometry motions are affine displacements and the displacements which send geodesics into geodesics without preserving the arc are projective.[4] In these cases the equations are obtained by replacing Γ^i_{jk} by the Christoffel symbols of the second kind.

The question of the existence of solutions of (3.6) or (3.8), together with further developments of this theory, will be presented in a later paper.

[1] Eisenhart, L. P., and Veblen, O., these PROCEEDINGS, **8,** 1922, p. 21.

[2] These PROCEEDINGS, **11,** 1925, p. 200.

[3] Eisenhart, L. P., *Riemannian Geometry,* **1926,** p. 223.

[4] Eisenhart, L. P., *loc. cit.*, pp. 228, 233.